THE RADIUS OF INJECTIVITY OF LOCAL RING Q–HOMEOMORPHISMS

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September 18, 2012

Abstract

The paper is devoted to the study of mappings with non–bounded characteristics of quasiconformality. The analog of the theorem about radius injectivity of locally quasiconformal mappings was proved for some class of mappings. There are found sharp conditions under which the so called local Q–homeomorphisms are injective in some neighborhood of a fixed point.

2010 Mathematics Subject Classification: Primary 30C65; Secondary 30C62

1 Introduction

Here are some definitions. Everywhere below, D is a domain in $\mathbb{R}^n$, $n \geq 2$, m be a measure of Lebesgue in $\mathbb{R}^n$, and $\mathrm{dist}\,(A, B)$ is the Euclidean distance between the sets A and B in $\mathbb{R}^n$. The notation $f : D \to \mathbb{R}^n$ assumes that f is continuous on its domain. In what follows (x, y) denotes the standard scalar multiplication of the vectors $x, y \in \mathbb{R}^n$, $\mathrm{diam}\, A$ is Euclidean diameter of the set $A \subset \mathbb{R}^n$,

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}\ , \quad \mathbb{B}^n := B(0, 1)\,,$$

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, \quad \mathbb{S}^{n-1} := S(0, 1)\,,$$

ω_{n-1} denotes the square of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, Ω_n is a volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^n$ is said to a *local homeomorphism* if for every $x_0 \in D$ there is a number $\delta > 0$ such that a mapping $f|_{B(x_0,\delta)}$ to be a homeomorphism.

Recall that a mapping $f : D \to \mathbb{R}^n$ is said to be *a mapping with bounded distortion*, if the following conditions hold:
1) $f \in W^{1,n}_{loc}$,
2) a Jacobian $J(x, f) := \det f'(x)$ of the mapping f at the point $x \in D$ preserves the sign almost everywhere in D,
3) $\|f'(x)\|^n \leq K \cdot |J(x, f)|$ at a.e. $x \in D$ and some constant $K < \infty$, where

$$\|f'(x)\| := \sup_{h \in \mathbb{R}^n : |h| = 1} |f'(x)h|\,,$$

see., e.g., § 3 Ch. I in [5], or definition 2.1 of the section 2 Ch. I in [6]. In this case we also say that f is *K–quasiregular,* where K is from condition 3) meaning above.

The following result was proved in the work [3] by O. Martio, S. Rickman and J. Väisälä, see Theorem 2.3 in [3] or Theorem 3.4 of Ch. III in [6], see also the paper [1].

Statement 1. *If $n \geq 3$ and $f : \mathbb{B}^n \to \mathbb{R}^n$ is a K–quasiregular local homeomorphism, then f is injective in a ball $B(0, \psi(n, K))$, where ψ is a positive number depending only on n and K.*

A goal of the present paper is a proof of the analog of the Statement 1 for more general classes of mappings of ring Q–homeomorphisms. To introduce this class of the mappings, we give some definitions.

A curve γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an open, closed or half–open interval in $\mathbb{R}$. Given a family Γ of paths γ in $\mathbb{R}^n$, $n \geq 2$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm} \Gamma$, if

$$\int\limits_{\gamma} \rho(x)\, |dx| \geq 1$$

for each $\gamma \in \Gamma$. The *modulus* of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\, dm(x)\, .$$

Given a domain D and two sets E and F in $\overline{\mathbb{R}^n}$, $n \geq 2$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$.

Let D be a domain in $\mathbb{R}^n$, $Q : D \to [0, \infty]$ be a (Lebesgue) measurable function. Set

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\, .$$

We say that a mapping $\overline{\mathbb{R}^n}$ is a *ring Q–mapping at a point* $x_0 \in D$ if

$$M\left(f(\Gamma(S(x_0, r_1), S(x_0, r_2), A(x_0, r_1, r_2)))\right) \leq \int\limits_{A(x_0,r_1,r_2)} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x) \qquad (1.1)$$

for every ring $A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < r_0 = \operatorname{dist}(x_0, \partial D)$, and for every Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r) dr \geq 1\, .$$

If the condition (1.1) holds at every point $x_0 \in D$, then we also say that f is a ring Q–mapping in the domain D, see section 7 in [4].

In what follows $q_{x_0}(r)$ denotes the integral average of $Q(x)$ under the sphere $|x - x_0| = r$,

$$q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\, dS\, , \qquad (1.2)$$

where dS is element of the square of the surface S.

One of the main results of the paper is following.

Theorem 1.1. *Let $n \geq 3$ and $f : \mathbb{B}^n \to \mathbb{R}^n$ is a local ring Q–homeomorphism at the point $x_0 = 0$, such that $Q \in L^1_{loc}(\mathbb{B}^n)$ and*

$$\int\limits_0^1 \frac{dt}{tq_0^{1/(n-1)}(t)} = \infty\,. \tag{1.3}$$

Then f is injective in a ball $B\left(0, \delta(n, Q)\right)$, where δ is a positive number depending only on n and function Q. From other hand, the condition (1.3) is precise, in fact, for every $\delta > 0$ and every $Q \in L^1_{loc}(\mathbb{B}^n)$ with $Q(x) \geq 1$ a.e. and

$$\int\limits_0^1 \frac{dt}{tq_0^{1/(n-1)}(t)} < \infty \tag{1.4}$$

there exists a mapping $f = f_Q : \mathbb{B}^n \to \mathbb{R}^n$ which is local ring Q–homeomorphism at the point $x_0 = 0$ and which is not injective in $B(0, \delta)$.

2 The main Lemma

A set $Q \subset \overline{\mathbb{R}^n}$ is said to be *relatively locally connected* if every point in $\overline{Q}$ has arbitrary small neighborhoods U such that $U \cap Q$ is connected.

We also need following statements, see Lemma III.3.1–Lemma III.3.3 in [6].

Proposition 2.1. *Let $f : G \to \overline{\mathbb{R}^n}$ be a local homeomorphism, let Q be a simply connected and locally pathwise connected set in $\overline{\mathbb{R}^n}$, and let P be a component of $f^{-1}(Q)$ such that $\overline{P} \subset G$. Then f maps P homeomorphically onto Q. If, in addition, Q is relatively locally connected, f maps $\overline{P}$ homeomorphically onto $\overline{Q}$.*

Proposition 2.2. *Let $f : G \to \overline{\mathbb{R}^n}$ be a local homeomorphism and let F be a compact set in G such that f_F is injective. Then f is injective in a neighborhood of F.*

Proposition 2.3. *Let $f : G \to \overline{\mathbb{R}^n}$ be a local homeomorphism, let $A, B \subset G$, and let f be homeomorphic in A and B. If $A \cap B \neq \varnothing$ and $f(A) \cap f(B)$ is connected, then f is homeomorphic in $A \cup B$.*

Finally, we need the following statement of P. Koskela, J, Onninen and K. Rajala, see Lemma 3.1 in [1].

Proposition 2.4. *Let $n \geq 1$ and $r > 0$. Let $a \neq b$, $a, b \in S(0, r)$. Then there exists a point $p = p(a, b) \in B(0, r)$ such that for every $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$ either*

$$0, b \in B(p, t) \quad and \quad a \notin B(p, t)$$

or

$$a, b \in B(p, t) \quad and \quad 0 \notin B(p, t)\,.$$

The following Lemma plays the main role in the following.

Lemma 2.1. *Let $n \geq 3$, $Q : \mathbb{B}^n \to [0, \infty]$ and $f : \mathbb{B}^n \to \mathbb{R}^n$ is a local ring Q–homeomorphism at the point $x_0 = 0$. Suppose that there exist a function $\psi : (0, 1) \to [0, \infty]$ and a constant $C = C(n, Q, \psi)$ such that*

$$0 < I(r_1, r_2) := \int\limits_{r_1}^{r_2} \psi(t)dt < \infty \quad \forall r_1, r_2 \in (0, 1) \tag{2.1}$$

and for some $\alpha > 0$

$$\int\limits_{r_1<|x|<r_2} Q(x)\psi(|x|)dm(x) \leq C \cdot I^{n-\alpha}(r_1, r_2)\,. \tag{2.2}$$

Let

$$I(0, 1) := \int\limits_{0}^{1} \psi(t)dt = \infty\,, \tag{2.3}$$

then f is injective in a ball $B\left(0, \delta(n, Q, \psi)\right)$, where δ is a positive number depending only on n, functions Q and ψ.

Proof. We may assume $f(0) = 0$. Let $r_0 = \sup\{r \in \mathbb{R} : r > 0, \overline{U(0, r)} \subset \mathbb{B}^n\}$, where $U(0, r)$ is the 0–component of $f^{-1}(B(0, r))$. Clearly $r_0 > 0$. Fix $r < r_0$ and set $U = U(0, r)$,

$$l^* = l^*(0, f, r) = \inf\{|z| : z \in \partial U\}\,,$$

$$L^* = L^*(0, f, r) = \sup\{|z| : z \in \partial U\}\,.$$

By Proposition 2.1, f maps $\overline{U}$ homeomorphically onto $\overline{B(0, r)}$. Thus f is injective in $B(0, l^*)$ and it suffices to find a lower bound for l^*.

Note that $L^* \to 1$ as $r \to r_0$. In fact, let $L^* \not\to 1$ as $r \to r_0$. Note that L^* is increasing function by r and, consequently, the limit of L^* as $r \to r_0$ exists. Now $L^* \to \varepsilon_0$ as $r \to r_0$, $\varepsilon_0 \in (0, 1)$. Then $\overline{U(0, r)} \subset B(0, \varepsilon_0)$ for all $r \in (0, r_0)$. Note that $B(0, r_0) \subset f(B(0, \varepsilon_0))$. Now $B(0, r_0) \subset f(B(0, \varepsilon_0))$ and hence there exists $r' > r_0$ such that $\overline{U(0, r')} \subset \mathbb{B}^n$ that contradicts to the definition of r_0. The above contradiction shows that $L^* \to 1$ as $r \to r_0$ that we need to prove.

Pick x and $y \in \partial U$ such that $|x| = L^*$ and $|y| = l^*$. Note that, by the definition of U, $f(x)$, $f(y) \in S(0, r)$. By Proposition 2.4 there exists a point $p \in B(0, r)$ such that, for every $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$, $f(x) \in B(p, t)$ and either $0 \in B(p, t)$ and $f(y) \notin B(p, t)$, or $0 \notin B(p, t)$ and $f(y) \in B(p, t)$. Fix such a t. Note that $0, f(y)$ and $f(x) \in \overline{f(B(0, l^*))}$ and, consequently, $f(B(0, l^*)) \cap B(p, t) \neq \varnothing \neq f(B(0, l^*)) \setminus B(p, t)$. Since $f(B(0, l^*))$ is connected, this implies that there exists a point $z_t \in S(p, t) \cap f(B(0, l^*))$, see Theorem 1.I.46.5 of [2].

Let z_t^* be the unique point in $f^{-1}(z_t) \cap B(0, l^*)$. Let $C_t(\varphi) \subset S(p, t)$ be the spherical cap with center z_t and opening angle φ,

$$C_t(\varphi) = \{y \in \mathbb{R}^n : |y - p| = t, (z_t - p, y - p) > t^2\varphi\}\,.$$

Let φ_t be the supremum of all φ for which the z_t^*–component of $f^{-1}(C_t(\varphi))$ gets mapped homeomorphically onto $C_t(\varphi)$. Let $C_t = C_t(\varphi_t)$ and let C_t^* be the z_t^*–component of $f^{-1}(C_t)$. We claim that C_t^* meets $S(0, L^*)$.

Suppose this is not true. Since C_t^* is connected and $C_t^* \cap B(0, L^*) \neq \varnothing$, this implies that $C_t^* \subset B(0, L^*)$, see Theorem 1.I.46.5 in [2]. Remark that, in this case, C_t^* is a compact subset of U and by Proposition 2.1 f maps $\overline{C_t^*}$ homeomorphically onto $\overline{C_t}$. By Proposition 2.2 f is injective in a neighborhood of $\overline{C_t^*}$. Thus $\varphi_t = \pi$, $\overline{C_t} = S(p, t)$ and $\overline{C_t^*}$ is a topological $(n-1)$–sphere. Note that bounded component D of $\mathbb{R}^n \setminus \overline{C_t^*}$ contained in $B(0, L^*)$. Now $f(D)$ is a compact subset of $f(\mathbb{B}^n)$ and, since the mapping f is open, $\partial f(D) \subset f(\partial D)$.

Remark that $f(D) \subset B(p, t)$. In fact, let $f(D) \not\subset B(p, t)$, then there exists $y \in f(D) \setminus \overline{B(p, t)}$. Now we have $\left(f(\mathbb{B}^n) \setminus \overline{B(p, t)}\right) \cap f(D) \neq \varnothing$ and, since $f(D)$ is compact subdomain of $f(\mathbb{B}^n)$, $\left(f(\mathbb{B}^n) \setminus \overline{B(p, t)}\right) \setminus f(D) \neq \varnothing$. Since $f(\mathbb{B}^n) \setminus \overline{B(p, t)}$ is connected, this implies that there exists $z \in \partial f(D) \cap \left(f(\mathbb{B}^n) \setminus \overline{B(p, t)}\right)$, see Theorem 1.I.46.5 of [2], that contradicts to the inclusion $\partial f(D) \subset B(p, t)$. Now $f(D) \subset B(p, t)$. Remark that $B(p, t) \subset f(D)$. Indeed, let there exists $a \in B(p, t) \setminus f(D)$. Since $B(p, t)$ is connected and $B(p, t) \cap f(D) \neq \varnothing$ this implies that $\partial f(D) \cap B(p, t) \neq \varnothing$, see Theorem 1.I.46.5 of [2]. The last relation contradicts to the inclusion $\partial f(D) \subset B(p, t)$. Thus $f(D) = B(p, t)$. By the definition, D is a component of $f^{-1}(B(p, t))$. By Proposition 2.1 f maps $\overline{D}$ onto $\overline{B(p, t)}$ homeomorphically.

Since $z_t^* \in \overline{C_t^*} \cap U$, $\overline{D} \cap \overline{U} \neq \varnothing$. Since f maps $\overline{U}$ homeomorphically onto $\overline{B(0, r)}$, f is injective in $\overline{U} \cup \overline{D}$. This is impossible, because in view of the equality $f(D) = B(p, t)$ and that $f(x) \in B(p, t)$ there exists a point $x_1 \neq x$, $x_1 \in D$, such that $f(x_1) = f(x)$. Thus C_t^* meets $S(0, L^*)$.

Let $k_t^* \in C_t^* \cap S(0, L^*)$ and $k_t = f(k_t^*)$. Let Γ_t' be the family of all curves connecting k_t and z_t in C_t. Moreover, let Γ' be the union of the curve families Γ_t', $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$. Denote by f_t the restriction of f to C_t^*. Then f_t maps C_t^* homeomorphically onto C_t. Furthermore, denote

$$\Gamma = \bigcup_{t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)} \left\{f_t^{-1} \circ \gamma : \gamma \in \Gamma_t'\right\} .$$

Since for every $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$, $z_t^* \in B(0, l^*)$ and $k_t \in S(0, L^*)$, be the definition of ring Q–mapping we have

$$M(f(\Gamma(S(0, l^*), S(0, L^*), A(0, l^*, L^*)))) \le \int\limits_{A(0, l^*, L^*)} Q(x) \cdot \eta^n(|x|) dm(x) \qquad (2.4)$$

for every function $\eta : (l^*, L^*) \to [0, \infty]$ with

$$\int\limits_{l^*}^{L^*} \eta(r) dr \ge 1 .$$

Setting $\eta(t) = \psi(t)/I(l^*, L^*)$, where ψ is the function from the condition of Lemma, we observe that η satisfies the above condition. Now from (2.2) and (2.4) we obtain that

$$M(\Gamma') = M(f(\Gamma(S(0, l^*), S(0, L^*), A(0, l^*, L^*)))) \le$$

$$\leq \int\limits_{A(0,l^*,L^*)} Q(x)\cdot\eta^n(|x|)dm(x) \leq C/I^\alpha(l^*,L^*)\,. \tag{2.5}$$

On other hand, by [8], Theorem 10.2,

$$\int\limits_{S(p,t)} \rho^n(x)dS \geq \frac{C_n}{t} \tag{2.6}$$

for every ρ for which $\int\limits_\gamma \geq 1$ for every $\gamma \in \Gamma'_t$. The integration of (2.6) over t yields

$$M(\Gamma') \geq C_n\,. \tag{2.7}$$

We obtain from (2.5) and (2.7) that

$$C_n \leq C/I^\alpha(l^*,L^*) \leq C/I^\alpha(l^*(0,f,r_0),L^*(0,f,r)) \tag{2.8}$$

because $I(\varepsilon_1,\varepsilon_2) > I(\varepsilon_3,\varepsilon_2)$ as $\varepsilon_3 > \varepsilon_1$. Letting into the limit as $r \to r_0$ in (2.8), we have

$$C_n \leq C/I^\alpha(l^*(0,f,r_0),1)\,. \tag{2.9}$$

First of all, from the (2.9) follows that $I(\varepsilon,1) < \infty$ for every $\varepsilon \in (0,1)$. Follow, let $l^*(0,f,r_0) \to 0$, then it follows from (2.3) that the right hand of (2.9) tends to zero, that contradicts to (2.9). Thus, $l^*(0,f,r_0) \geq \delta$ for all such f. The proof is complete. $\Box$

3 Proof of the main result

The following statement would be very useful, see Theorem 3.15 in [7].

Proposition 3.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [0,\infty]$ a locally integrable measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a ring Q–homeomorphism at a point x_0 if and only if for every $0 < r_1 < r_2 < r_0 = \operatorname{dist}(x_0,\partial D)$,*

$$M\left(\Gamma\left(f(S_1), f(S_2), f(D)\right)\right) \leq \frac{\omega_{n-1}}{I^{n-1}}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $q_{x_0}(r)$ is the average of $Q(x)$ over the sphere $|x-x_0| = r$, $S_j = \{x \in \mathbb{R}^n : |x-x_0| = r_j\}$, $j = 1,2$, and

$$I = I(r_1,r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\,.$$

Proof of Theorem 1.1. Given $0 < r_1 < r_2 < r_0 = 1$ consider the function

$$\psi(t) \quad = \quad \begin{cases} 1/[tq_0^{\frac{1}{n-1}}(t)]\,, & t \in (r_1,r_2)\,,\\ 0\,, & t \notin (r_1,r_2)\,, \end{cases} \tag{3.1}$$

Note that ψ satisfies all the conditions of Lemma 2.1, in particular, $\int\limits_{r_1}^{r_2} \frac{dt}{tq_0^{1/(n-1)}(t)} < \infty$ by Theorem 1 in [9], and by Fubini theorem, $\int\limits_{r_1<|x|<r_2} Q(x)\cdot\psi^n(|x|)\,dm(x) = \omega_{n-1}\cdot I(r_1,r_2)$. Now the first part of the Theorem follows from Lemma 2.1.

To prove second part of the Theorem, we take $\delta>0$ and some function $Q\in L^1_{loc}(\mathbb{B}^n)$ satisfying (1.4). Set

$$f(x)=\frac{x}{|x|}\rho(|x|)\,,$$

where

$$\rho(r)=\exp\left\{-\int\limits_r^1\frac{dt}{t\widetilde{q}_0^{1/(n-1)}(t)}\right\},\qquad \widetilde{q}_0(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x|=r}\widetilde{Q}(x)\,dS\,,$$

$$\widetilde{Q}(x)\quad=\quad\begin{cases} Q(x), & |x|>\delta\,,\\ 1\,, & |x|\le\delta\,.\end{cases}$$

Note that a mapping f is a ring $\widetilde{Q}$–homeomorphism at $x_0=0$. In fact, we have $f(S(0,r))=S(0,R)$, where $R:=\exp\left\{-\int\limits_r^1\frac{dt}{t\widetilde{q}_0^{1/(n-1)}(t)}\right\}$. Now

$$f(\Gamma(S(0,r_1),S(0,r_2),A(0,r_1,r_2)))=\Gamma(S(0,R_1),S(0,R_2),A(0,R_1,R_2))\,,$$

where $R_i:=\exp\left\{-\int\limits_{r_i}^1\frac{dt}{t\widetilde{q}_0^{1/(n-1)}(t)}\right\}$, $i=1,2$. Now by 7.5 in [8],

$$M(f(\Gamma(S(0,r_1),S(0,r_2),A(0,r_1,r_2))))=\frac{\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2}\frac{dt}{t\widetilde{q}_0^{1/(n-1)}(t)}\right)^{n-1}}\,.$$

Now f is a ring $\widetilde{Q}$–homeomorphism at the point $x_0=0$ by Proposition 3.1. Note that under $\delta\to0$ the image $f(B(0,\delta))$ includes the ball $B(0,\sigma)$, where σ does not depend on δ. Now we map the ball $B(0,\sigma)$ by some map g, which is K–quasiregular and local homeomorphism for some $K\ge1$, but not injective in $B(0,\sigma)$; for instance, let g is a winding map, whose axes of rotation does not contain a ball $f(B(0,1))$, see 5.1 in [5]. Remark that K does not depend on δ. Now we construct a local ring $K\cdot Q(x)$–homeomorphism f_2 at zero, $f_2=g\circ f$, which is not injective in $B(0,\delta)$. Since Q is arbitrary function with $Q\ge1$ satisfying (1.4), we can replace Q on the Q/K in the start of the second part of the proof. So, we obtain a local ring $Q(x)$–homeomorphism with the properties meaning above. The proof is complete. $\square$

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